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A class of solutions of the Ricci and Einstein equations

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Abstract

We consider the pseudo-Euclidean space (R^n, g), with $n \geq 3$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$ and tensors of the form $T = \sum_i f_i(x_k)\epsilon_i \mathrm{d}x_i^2$ for a fixed k, $1 \leq k \leq n$. We provide necessary and sufficient conditions for such a tensor to admit metrics $\bar{g}$, conformal to g, that solve the Ricci equation or the Einstein equation. The solution to this problem is given explicitly and it depends on an arbitrary differentiable function of one variable. Similar problems are considered for locally conformally flat manifolds. Examples are provided of complete metrics on R^n, whose Ricci curvature is negative. Complete metrics are also given on the cylinder or on the n-dimensional torus, that solve the Ricci equation or the Einstein equation. Examples of metrics with positive Ricci curvatures are given on half-spaces of R^n.

MSC: 53C21; 53C50; 53C80

Keywords: Ricci tensor; Conformal metric; Ricci equation; Einstein equation

1. Introduction

In this paper, we consider a special class of symmetric tensors T on a pseudo-Euclidean space and we determine all metrics, conformal to the pseudo-Euclidean metric, whose Ricci tensor is the given tensor T. A similar question is considered for the Einstein equation. The theory is also extended to locally conformally flat manifolds.

Different aspects of the more general problem of finding a metric g whose Ricci tensor is a given second-order symmetric tensor T were considered in several papers. DeTurck in [2] showed that this problem has a local solution when T is a nonsingular tensor defined on a manifold M^n, $n \geq 3$. Cao and DeTurck [1] considered rotationally symmetric nonsingular tensors. For special classes of tensors T on the n-dimensional pseudo-Euclidean space and on the hyperbolic space, with $n \geq 3$, we obtained explicitly in [9,8] all metrics $\bar{g}$, conformal to the standard metric, such that $\mathrm{Ric}\,\bar{g} = T$. In [11,12], we considered the same problem for $T = fg$, on the pseudo-Euclidean space, the sphere and the hyperbolic space, with the usual metric g, where f is a differentiable function. For the two-dimensional case, existence and uniqueness results can be found in [3,5].

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With respect to the higher dimensional Einstein equation, we are looking for tensors T on an n-dimensional manifold, that admit metrics g such that $\mathrm{Ric}\, g - \frac{K}{2} g = T$, where K is the scalar curvature of g. When $n = 4$ and the metric is Lorentzian, then the equation is known as the Einstein field equation. DeTurck [4] considered the Cauchy problem for this equation with nonsingular tensors. When the tensor T represents several physical situations, this equation has been studied by several authors (see [6]). In [10], we considered this problem for special tensors and metrics conformal to the standard metric of the n-dimensional pseudo-Euclidean space, $n \geq 3$. In [11], we determined all tensors of the form $T = fg$, where f is a differentiable function, g is the standard metric on the sphere that admits a solution $\bar{g}$ conformal to g, for the Einstein equation. The analogous problems for the pseudo-Euclidean and the hyperbolic spaces were considered in [12].

In this paper, we extend these results to a more general class of tensors. More precisely, we consider (R^n, g), with $n \geq 3$, coordinates $x = (x_1, \ldots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$, where at least an ϵ_i is positive. We consider tensors of the form

$$T = \sum_i f_i(x_k)\epsilon_i \mathrm{d}x_i^2 \tag{1}$$

for a fixed k, $1 \leq k \leq n$, assuming that not all functions f_i are constant and not all f_i are equal.

We want to find $\bar{g} = \frac{1}{\varphi^2} g$, which is a solution for the Ricci equation or the Einstein equation. More precisely, we want to solve the following problems:

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g \\ \mathrm{Ric}\, \bar{g} = T. \end{cases} \tag{2}$$

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g \\ \mathrm{Ric}\, \bar{g} - \dfrac{\bar{K}}{2}\bar{g} = T. \end{cases} \tag{3}$$

We will show that the class of tensors of type (1), that solve problems (2) or (3) depend on an arbitrary differentiable function of one variable. With respect to the assumptions on the functions f_i of the tensor T, we observe that the cases when all f_i are constant or when they are all equal were considered in [9–12].

In Theorem 1.1 we provide necessary and sufficient conditions for solving (2) and the solutions are given explicitly. In Theorem 1.2, we obtain analogous results for the Einstein equation (3). We also extend the results to locally conformally flat manifolds. By applying the theory, we exhibit examples of complete metrics on R^n, whose Ricci curvature is negative. The existence of such metrics in any Riemannian manifold was proved in [7] by Lohkamp. Examples of complete metrics are also given on the cylinder or on the n-dimensional torus, that solve the Ricci equation or the Einstein equation. On half-spaces of R^n, we provide metrics with positive Ricci curvatures.

As a consequence of Theorem 1.1, we show that for certain functions $\bar{K}$, depending on one variable, there exist metrics $\bar{g}$, conformal to the pseudo-Euclidean metric g, whose scalar curvature is $\bar{K}$. Equivalently, we find C^∞ solutions for the equation

$$\frac{4(n-1)}{n-2}\Delta_g u + \bar{K} u^{\frac{n+2}{n-2}} = 0. \tag{4}$$

where Δ_g denotes the Laplacian in the pseudo-Euclidean metric g. We observe that in the Riemannian case there exist functions $\bar{K}$ whose corresponding metrics are complete on R^n. The metrics on R^n conformal to the Euclidean metric obtained from solutions of (4) provide solutions for the following problem: Given a differentiable function $\bar{K}$, on Riemannian manifold (M, g), is there a metric $\bar{g}$ conformal to g whose scalar curvature is $\bar{K}$? This problem has been studied by many authors. In particular, when $\bar{K}$ is constant, it is known as the Yamabe problem.

2. Main results

We will now state our main results. The proofs will be given in the following section.

Theorem 1.1. *Let (R^n, g), $n \geq 3$, be a pseudo-Euclidean space, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider the tensor $T = \sum_{i=1}^n f_i(x_k)\epsilon_i \mathrm{d}x_i^2$, for some fixed k. Assume that not all f_i are constant and that not all f_i are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that* Ric $\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ such that*

$$f_k(x_k) = \epsilon_k(n-1)U''(x_k) \tag{5}$$

$$f_j(x_k) = \epsilon_k[U''(x_k) - (n-2)(U'(x_k))^2] \quad \forall j \neq k. \tag{6}$$

$$\varphi = \mathrm{e}^{U(x_k)}. \tag{7}$$

We observe that the case of Theorem 1.1, when all functions f_i are constant, was considered in Theorems 1.3 and 1.4 of [9] and the case when all functions f_i are equal was investigated in [12].

Theorem 1.2. *Let (R^n, g), $n \geq 3$, be a pseudo-Euclidean space with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Consider $T = \sum_{i=1}^n f_i(x_k)\epsilon_k \mathrm{d}x_i^2$ for some fixed k. Assume that not all f_i are constant and that not all f_i are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that* Ric $\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ such that*

$$f_k(x_k) = \epsilon_k \frac{(n-1)(n-2)}{2}(U'(x_k))^2 \tag{8}$$

$$f_j(x_k) = \epsilon_k(n-2)\left(\frac{(n-3)}{2}(U'(x_k))^2 - U''(x_k)\right) \quad \forall j \neq k. \tag{9}$$

$$\varphi = \mathrm{e}^{U(x_k)}. \tag{10}$$

We observe that the case of Theorem 1.2, when all functions f_i are constant, was considered in Theorems 3 and 4 of [10] and the case when all functions f_i are equal was studied in [12].

Corollary 1.3. *If (R^n, g) is the Euclidean space and $\varphi(x_k) \leq C$ for some constant $C > 0$, then the metrics given by Theorems 1.1 and 1.2 are complete on R^n.*

Example 1.4. As a direct consequence of Theorem 1.1, Theorem 1.2 and Corollary 1.3 we get the following examples, where we are considering (R^n, g), $n \geq 3$, the pseudo-Euclidean space with coordinates $(x_1, \ldots, x_n)$ such that $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$.

(a) Consider the function $U = -x_k^m$, for some fixed k, where m is an even number and the tensor

$$T = -m x_k^{m-2}\left\{(m-1)(n-1)\mathrm{d}x_k^2 + \epsilon_k\left[m - 1 + (n-2)m x_k^m\right]\sum_{j\neq k}\epsilon_j \mathrm{d}x_j^2\right\},$$

determined as in Theorem 1.1. We observe that although this tensor is singular on the hyperplane $x_k = 0$, there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that Ric $\bar{g} = T$, globally defined on R^n with $\varphi = \exp(-x_k^m)$. Moreover, it follows from Corollary 1.3, that in the Euclidean case, the metric $\bar{g}$ is a complete metric on R^n, whose Ricci curvature is negative.

(b) Consider the periodic function $U = \sin x_k$ for some fixed k. Then the tensor

$$T = \sum_{j=1}^n \{\epsilon_k\epsilon_j(1-\delta_{jk})[-\sin x_k - (n-2)\cos^2 x_k] - \delta_{jk}(n-1)\sin x_k\}\mathrm{d}x_j^2$$

admits the metric $\bar{g} = g/\mathrm{e}^{2\sin x_k}$ whose Ricci tensor is T. The scalar curvature is given by

$$\bar{K} = -\epsilon_k(n-1)\mathrm{e}^{2\sin x_k}[2\sin x_k + (n-2)\cos^2 x_k]$$

and it has positive and negative values. Observe that $\bar{g}$ is periodic in all variables and when $\epsilon_j = 1$, it can be considered to be a complete metric on a cylinder or an n-dimensional torus.

(c) We observe that, as a consequence of Theorem 1.2, the periodic function $U = \sin x_k$, for some fixed k, determines a tensor

$$T = \sum_{j=1}^{n} \epsilon_k \epsilon_j \frac{n-2}{2} \{(1-\delta_{jk})[2\sin x_k + (n-3)\cos^2 x_k] + \delta_{jk}(n-1)\cos^2 x_k\} \mathrm{d}x_j^2$$

which admits a solution $\bar{g} = g/\mathrm{e}^{2\sin x_k}$ for the Einstein equation. In the Euclidean case, this can be considered as an example of a tensor defined on a torus, that admits a solution for the Einstein equation.

Corollary 1.5. *Let (R^n, g) be a pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij}\epsilon_i$, $\epsilon_i = \pm 1$. Let $\bar{K} : R^n \to R$ be given by*

$$\bar{K} = \epsilon_k (n-1) \mathrm{e}^{2U(x_k)} [2U'' - (n-2)(U')^2] \tag{11}$$

for some x_k, where $U(x_k)$ is a differentiable function. Then the differential equation

$$\frac{4(n-1)}{n-2} \Delta_g u + \bar{K} u^{\frac{n+2}{n-2}} = 0 \tag{12}$$

where Δ_g denotes the Laplacian in the metric g, has a solution, globally defined on R^n, given by

$$u = \mathrm{e}^{-\frac{(n-2)}{2} U(x_k)}. \tag{13}$$

The geometric interpretation of the corollary above is the following:

Corollary 1.6. *Let (R^n, g) be a pseudo-Euclidean space, $n \geq 3$, and $\bar{K}$ a function given by (11). Then there exists a metric $\bar{g} = u^{\frac{4}{n-2}} g$, where u is given by (13), whose scalar curvature is $\bar{K}$. In particular, if (R^n, g) is the Euclidean space and u is a bounded function then $\bar{g}$ is a complete metric.*

Example 1.7. As a direct consequence of Corollary 1.5 we have the following example, locally conformal to the Euclidean space (R^n, g). Consider the function

$$U(x_1) = -\frac{1}{n-1} \log \left(-\frac{n-1}{2} x_1 + b \right)^2$$

defined on the half-space of R^n where $(n-1)x_1 - 2b < 0$ and the scalar curvature given by (11). Then the metric $\bar{g} = g/\mathrm{e}^{2U}$ has positive scalar and Ricci curvatures.

We now consider a Riemannian manifold locally conformally flat (M^n, g); then one can consider problems (2) and (3) for any neighborhood $V \subset M$ such that there are local coordinates $(x_1, \ldots, x_n)$ with $g_{ij} = \delta_{ij}/F^2$, where F is a nonvanishing differentiable function on V. It is easy to see that the following results hold.

Theorem 1.8. *Let (M^n, g), $n \geq 3$, be a Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $(x_1, \ldots, x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider the tensor $T = \sum_{i=1}^{n} f_i(x_k) \mathrm{d}x_i^2$ for some fixed k. Assume that not all f_i are constant and not all of them are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that* Ric $\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ on V such that*

$$f_k(x_k) = (n-1)U''(x_k) \tag{14}$$

$$f_j(x_k) = [U''(x_k) - (n-2)(U'(x_k))^2] \quad \forall j \neq k. \tag{15}$$

$$\varphi = \frac{1}{F} \mathrm{e}^{U(x_k)}.$$

Theorem 1.9. *Let (M^n, g), $n \geq 3$, be a Riemannian manifold, locally conformally flat. Let V be an open subset of M with coordinates $(x_1, \ldots, x_n)$ such that $g_{ij} = \delta_{ij}/F^2$. Consider the tensor $T = \sum_{i=1}^{n} f_i(x_k) \mathrm{d}x_i^2$ for some fixed k.*

Assume that not all f_i are constant and not all are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g$ such that $\mathrm{Ric}\,\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ such that*

$$f_k(x_k) = \frac{(n-1)(n-2)}{2}(U'(x_k))^2 \tag{16}$$

$$f_j(x_k) = (n-2)\left(\frac{(n-3)}{2}(U'(x_k))^2 - U''(x_k)\right) \quad \forall j \neq k. \tag{17}$$

$$\varphi = \frac{1}{F}\mathrm{e}^{U(x_k)}.$$

As a consequence of Theorems 1.8 and 1.9, we have the following results for the hyperbolic space.

Corollary 1.10. *Let (R^n_+, g^*), $n \geq 3$, with $g^*_{ij} = \delta_{ij}/x_n^2$ be the hyperbolic space. Let $T = \sum_{i=1}^n f_i(x_k)\mathrm{d}x_i^2$ for some fixed k. Assume that not all f_i are constant and not all of them are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g^*$ such that* $\mathrm{Ric}\,\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ on R^n_+ such that $f_k(x_k)$ and $f_j(x_k)$ are given by (14), (15) and*

$$\varphi = \frac{1}{x_n}\mathrm{e}^{U(x_k)}.$$

Corollary 1.11. *Let (R^n_+, g^*), $n \geq 3$, be the hyperbolic space with coordinates $(x_1, \ldots, x_n)$ such that $g^*_{ij} = \delta_{ij}/x_n^2$. Consider the tensor $T = \sum_{i=1}^n f_i(x_k)\mathrm{d}x_i^2$ for some fixed k. Assume that not all f_i are constant and not all of them are equal. Then there exists $\bar{g} = \frac{1}{\varphi^2} g^*$ such that* $\mathrm{Ric}\,\bar{g} - \frac{\bar{K}}{2}\bar{g} = T$ *if, and only if, there exists a differentiable function $U(x_k)$ on R^n_+ such that $f_k(x_k)$ and $f_j(x_k)$ are given by (16), (17) and*

$$\varphi = \frac{1}{x_n}\mathrm{e}^{U(x_k)}.$$

Corollary 1.12. *Let (R^n_+, g^*), $n \geq 3$, be the hyperbolic space and $\varphi(x_k) \leq C$ for some constant $C > 0$; then the metrics given by Corollaries 1.10 and 1.11 are complete on R^n_+.*

Example 1.13. As a direct consequence of the previous result, we have the following example. Let (R^n_+, g^*), $n \geq 3$, be the hyperbolic space and $\varphi(x) = \mathrm{e}^{-x_n^m}$, where m is an even number. Then $\bar{g} = g^*/\varphi^2$ is a complete metric on R^n_+ and its Ricci curvature is negative.

We observe that there are similar results for manifolds that are locally conformal to the pseudo-Euclidean space.

4. Proof of the main results

Proof of Theorem 1.1. Since $\mathrm{Ric}\, g = 0$, we have that $\bar{g} = \frac{1}{\varphi^2} g$ is such that $\mathrm{Ric}\,\bar{g} = T$ if, and only if,

$$T = \mathrm{Ric}\,\bar{g} = \frac{1}{\varphi^2}\{(n-2)\varphi \mathrm{Hess}_g(\varphi) + [\varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2]g\}. \tag{18}$$

This is equivalent to the following system of equations:

$$\varphi_{,x_i x_j} = 0, \quad \forall i \neq j, \tag{19}$$

where $\varphi_{,x_i x_j}$ denotes the second-order derivative of φ with respect to $x_i x_j$ and

$$\epsilon_i f_i(x_k) = (n-2)\frac{\varphi_{,x_i x_i}}{\varphi} + \epsilon_i \frac{\Delta_g \varphi}{\varphi} - \epsilon_i (n-1)\frac{|\nabla_g \varphi|^2}{\varphi^2} \quad \forall i. \tag{20}$$

From (19), we get $\varphi = \sum_{i=1}^{n} \varphi_i(x_i)$, which substituted into (20) gives

$$f_i(x_k) = \epsilon_i (n-2)\frac{\varphi_i''}{\varphi} + \frac{\Delta_g \varphi}{\varphi} - (n-1)\frac{|\nabla_g \varphi|^2}{\varphi^2}, \quad \forall i. \tag{21}$$

We will show that φ depends only on x_k and that all functions f_i for $i \neq k$ are equal.

As a consequence of (21), we get

$$\left[f_i(x_k) - f_j(x_k)\right]\varphi = (n-2)[\epsilon_i \varphi_i''(x_i) - \epsilon_j \varphi_j''(x_j)] \quad \forall i \neq j. \tag{22}$$

By hypothesis, there exists $i_0 \neq k$ such that $f_{i_0} \neq f_k$. Then taking the derivative of $f_{i_0} - f_k$, given by (22), with respect to x_j, for $j \neq i_0$ and $j \neq k$, we get that $\varphi_{,x_j} = 0$. Hence $\varphi = \varphi_{i_0}(x_{i_0}) + \varphi_k(x_k)$.

If for all j, $j \neq i_0$ and $j \neq k$, we have $f_k(x_k) - f_j(x_k) = 0$, then from (22) we get that $\varphi_k''(x_k) = 0$. In this case, it follows from (21) that

$$f_k(x_k) = f_j(x_k) = \frac{\Delta_g \varphi}{\varphi} - (n-1)\frac{|\nabla_g \varphi|^2}{\varphi^2}.$$

Moreover, from $f_{i_0} - f_k$ given by (22), we get $\varphi_{i_0}''(x_{i_0}) \neq 0$ and

$$\frac{\varphi_{i_0}(x_{i_0}) + \varphi_k(x_k)}{\varphi_{i_0}''(x_{i_0})} = \frac{(n-2)\epsilon_{i_0}}{f_{i_0}(x_k) - f_k(x_k)}. \tag{23}$$

Taking the derivative of this equation with respect to x_k, since not all f_i are constant, we obtain that $\varphi_{i_0}''(x_{i_0}) = a \neq 0$, where a is a real constant. Moreover, taking the derivative of (23) with respect to x_{i_0}, we conclude that $a = 0$, which is a contradiction.

Therefore, there exists j_0, $j_0 \neq i_0$ and $j_0 \neq k$, such that $f_k(x_k) - f_{j_0}(x_k) \neq 0$. Then taking the derivative of (22) with respect to x_{i_0}, we get $\varphi_{i_0}'(x_{i_0}) = 0$. Hence φ depends only on x_k and it follows from (22) that

$$f_i(x_k) - f_j(x_k) = 0 \quad \forall i, j \; i \neq k \; j \neq k.$$

Moreover, from (21) we have

$$f_k = \epsilon_k (n-1)\left(\frac{\varphi''}{\varphi} - \frac{(\varphi')^2}{\varphi^2}\right), \tag{24}$$

$$f_j = \epsilon_k \left(\frac{\varphi''}{\varphi} - (n-1)\frac{(\varphi')^2}{\varphi^2}\right), \quad j \neq k. \tag{25}$$

Now we consider

$$\frac{\varphi'}{\varphi} = U'(x_k) \tag{26}$$

where $U(x_k)$ is a differentiable function. It follows from (24)–(26) that φ is given by (7) and that f_k and f_j are given by (5) and (6) respectively. The converse is a straightforward computation. □

Proof of Theorem 1.2. Since Ric $g = 0$ and

$$\bar{K} = (n-1)(2\varphi \Delta_g \varphi - n|\nabla_g \varphi|^2),$$

we have that $\bar{g} = \frac{1}{\varphi^2} g$ is such that Ric $\bar{g} - \bar{K}\bar{g}/2 = T$ if, and only if,

$$T = \frac{1}{\varphi^2}\left\{(n-2)\varphi \mathrm{Hess}_g(\varphi) + \left[-(n-2)\varphi \Delta_g \varphi + \frac{(n-1)(n-2)}{2}|\nabla_g \varphi|^2\right] g\right\}. \tag{27}$$

This is equivalent to the following system of equations:

$$\varphi_{,x_i x_j} = 0, \quad \forall i \neq j \tag{28}$$

and

$$\epsilon_i f_i(x_k) = (n-2)\left(\frac{\varphi_{,x_i x_i}}{\varphi} - \epsilon_i \frac{\Delta_g \varphi}{\varphi} + \epsilon_i (n-1)\frac{|\nabla_g \varphi|^2}{2\varphi^2}\right) \quad \forall i. \tag{29}$$

From (28), we get $\varphi = \sum_{i=1}^n \varphi_i(x_i)$, which substituted into (29) gives

$$f_i(x_k) = (n-2)\left(\epsilon_i \frac{\varphi_i''}{\varphi} - \frac{\Delta_g \varphi}{\varphi} + (n-1)\frac{|\nabla_g \varphi|^2}{2\varphi^2}\right), \quad \forall i. \tag{30}$$

As a consequence of this last equation, we get

$$\left[f_i(x_k) - f_j(x_k)\right]\varphi = (n-2)[\epsilon_i \varphi_i''(x_i) - \epsilon_j \varphi_j''(x_j)] \quad \forall i \neq j. \tag{31}$$

By hypothesis, there exists $i_0 \neq k$ such that $f_{i_0} \neq f_k$. Then taking the derivative of (31) with respect to x_j, for $j \neq i_0$ and $j \neq k$, we get that $\varphi_{,x_j} = 0$. Hence $\varphi = \varphi_{i_0}(x_{i_0}) + \varphi_k(x_k)$.

If for all j, $j \neq i_0$ and $j \neq x_k$, we have $f_k(x_k) - f_j(x_k) = 0$, then from (31) we get that $\varphi_k''(x_k) = 0$. In this case, we have

$$f_k(x_k) = f_j(x_k) = (n-2)\left(-\frac{\Delta_g \varphi}{\varphi} + (n-1)\frac{|\nabla_g \varphi|^2}{2\varphi^2}\right).$$

Moreover, from (31), we get $\varphi_{i_0}''(x_{i_0}) \neq 0$ and

$$\frac{\varphi_{i_0}(x_{i_0}) + \varphi_k(x_k)}{\varphi_{i_0}''(x_{i_0})} = \frac{(n-2)\epsilon_{i_0}}{f_{i_0}(x_k) - f_k(x_k)}. \tag{32}$$

Taking the derivative of this equation with respect to x_k, since not all f_i are constant, we obtain that $\varphi_{i_0}''(x_{i_0}) = a \neq 0$, where a is a real constant. Taking the derivative of (32) with respect to x_{i_0}, we conclude that $a = 0$, which is a contradiction.

Therefore, there exists j_0, $j_0 \neq i_0$ and $j_0 \neq k$, such that $f_k(x_k) - f_{j_0}(x_k) \neq 0$. Then taking the derivative of (31) with respect to x_{i_0}, we get $\varphi_{i_0}(x_{i_0}) = 0$. Hence φ depends only on x_k and it follows from (31) that

$$f_i(x_k) - f_j(x_k) = 0 \quad \forall i, j \text{ such that } i \neq k\ j \neq k.$$

Moreover, from (29) we have

$$f_k = \epsilon_k (n-1)(n-2)\frac{(\varphi')^2}{2\varphi^2}, \tag{33}$$

$$f_j = \epsilon_k (n-2)\left[-\frac{\varphi''}{\varphi} + (n-1)\frac{(\varphi')^2}{2\varphi^2}\right], \quad j \neq k. \tag{34}$$

Now we consider

$$\frac{\varphi'}{\varphi} = U'(x_k)$$

where $U(x_k)$ is a differentiable function. It follows that φ is given by (10) and from (33) and (34) we conclude that (8) and (9) hold. The converse is a straightforward computation. □

Proof of Corollary 1.3. Consider the Euclidean space (R^n, g), $n \geq 3$, and a metric $\bar{g}$ given by Theorem 1.1 or Theorem 1.2. If $\varphi(x_k) \leq C$, where $C > 0$, then the metric $\bar{g}$ is complete, since there exists a constant $m > 0$, such that for any vector $v \in R^n$, $|v|_{\bar{g}} \geq m|v|$. □

Proof of Corollary 1.5. It follows from (5) and (6) that for the metric $\bar{g}$ of Theorem 1.1, the scalar curvature is

$$\bar{K} = \epsilon_k (n-1) e^{2U}[2U'' - (n-2)(U')^2]. \tag{35}$$

Define the function $u^{\frac{-2}{n-2}} = e^U = \varphi$. Substituting for U and its derivatives in terms of u, we obtain (12). □

Proof of Corollary 1.6. This result follows immediately from the previous corollary, since finding a metric $\bar{g} = u^{\frac{4}{n-2}} g$ with scalar curvature $\bar{K}$ is equivalent to solving Eq. (12). □

For the proofs of Theorems 1.8 and 1.9, we consider the function $\tilde{\varphi} = \varphi F$. Then arguments similar to those of Theorems 1.1 and 1.2 complete the proofs.

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